Lifting and Separation Procedures for the Cut Polytope

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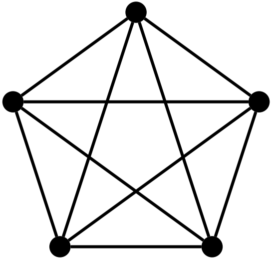
6th Conference of PhD Students in Computer Science
Szeged, July 3, 2008

Outline

Outline

Definition

Let $G = (V, E, c)$ be an undirected weighted graph.



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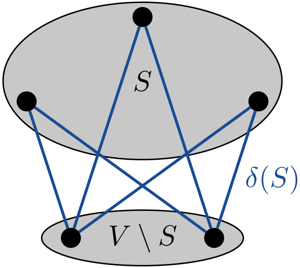
Any $S \subseteq V$ induces a set of edges $\delta(S)$ with exactly one endpoint in S. $\delta(S)$ is called a cut of G with shores S and $V \setminus S$.



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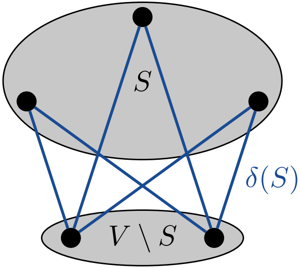
Finding a cut with maximum aggregate edge weight is known as max-cut problem.



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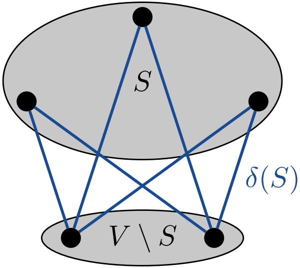
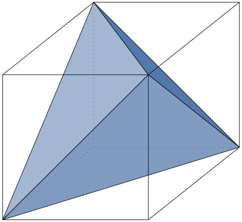


Applications

- quadratic +/–1 resp. 0/1 optimization,
- determining ground states of Ising spin glasses.

Cut polytope CUT(G)

Convex hull of the incidence vectors of all cuts of G.



$\mathrm{CUT}(K_3)$

Cycle polytope M(G)

Relaxation of the cut polytope with linear description

$$\begin{aligned} x(F) - x(C \setminus F) &\leq |F| - 1, && \text{for all } F \subseteq C, |F| \text{ odd}, \\ & && \text{for each cycle } C \text{ of } G, \\ x_e &\in [0,1], && e \in E. \end{aligned}$$

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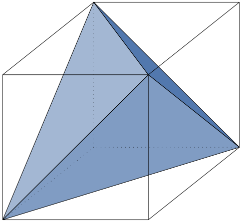

CUT(K_3)

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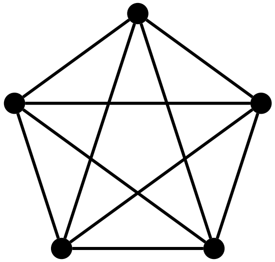
CUT(G) and M(G) have exactly the same integral points.

Algorithms

- Branch&Cut (possibly combined with Semidefinite Programming) as exact method,
- most techniques are associated with dense/complete graphs.



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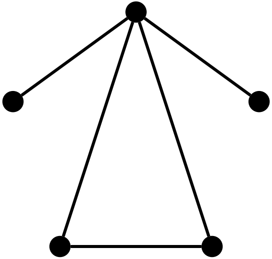

Handling sparse graphs

Algorithms

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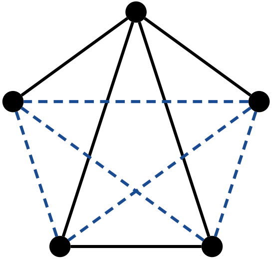


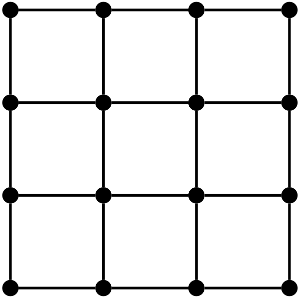
Handling sparse graphs

- **trivial approach**:
  artificial completion using edges with weight zero,
- **major drawback**:
  increase in number of variables/computational difficulty.

Outline

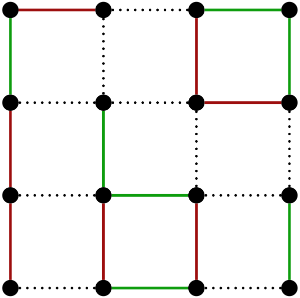
An example

(4×4)-grid with 16 nodes and 24 edges.



(4×4)-grid with 16 nodes and 24 edges.

W.r.t. a vector $z \in \mathsf{M}(G) \setminus \mathsf{CUT}(G)$ the edge set decomposes into:



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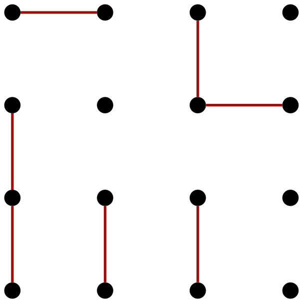

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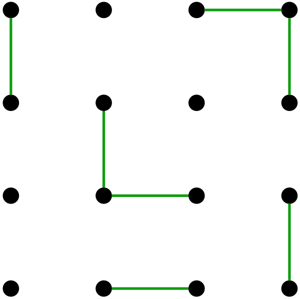
- 1-edges,



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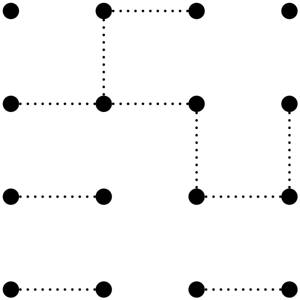
W.r.t. a vector $z \in \mathsf{M}(G) \setminus \mathsf{CUT}(G)$ the edge set decomposes into:

- 0-edges,
- 1-edges,
- fractional edges.



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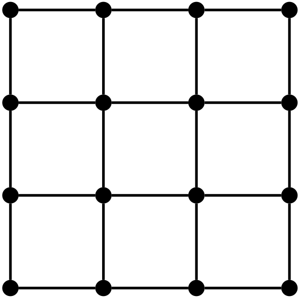
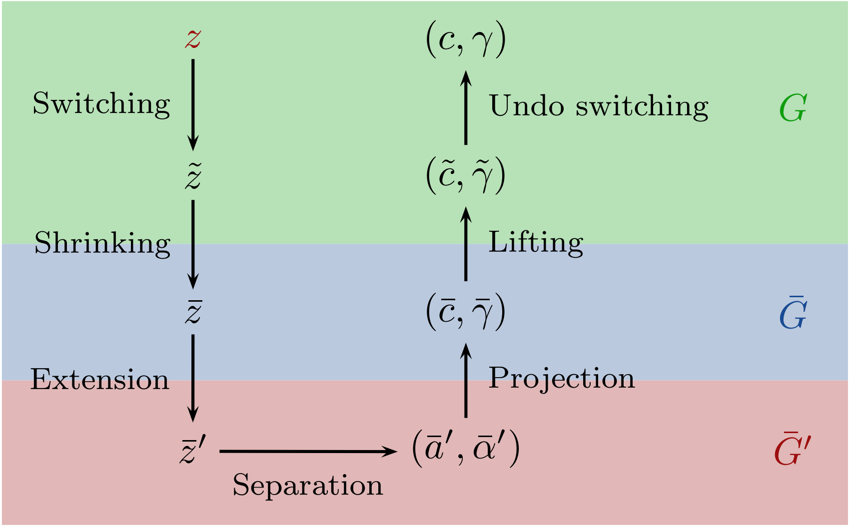


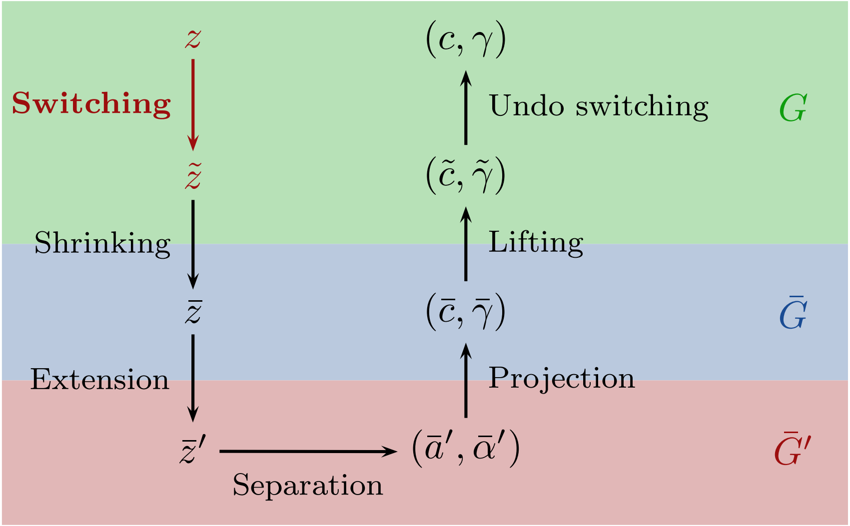
Artificial completion would require 96 additional edges.

Input: vector $z \in \mathsf{M}(G) \setminus \mathsf{CUT}(G)$.



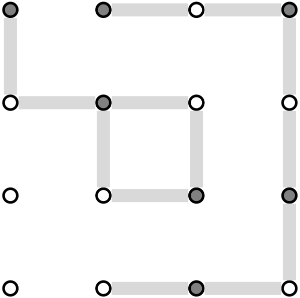
Outline of the shrinking approach

Transform 1-edges into 0-edges without affecting original 0-edges.



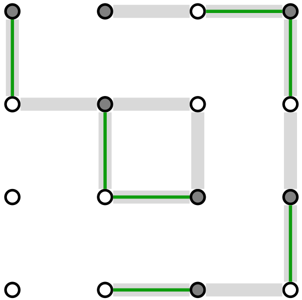
Switching

$z \in \mathrm{M}(G)$ implies existence of a cut that contains

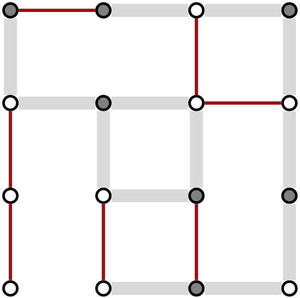


Switching

$z \in \mathrm{M}(G)$ implies existence of a cut that contains all 1-edges



$z \in \mathrm{M}(G)$ implies existence of a cut that contains all 1-edges but no 0-edges.



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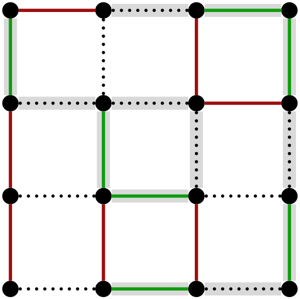
Switching z alongside this cut

- only affects cut edges,



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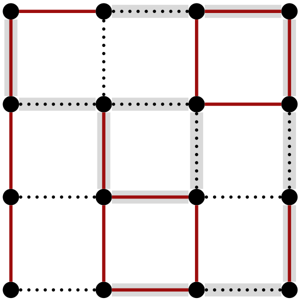

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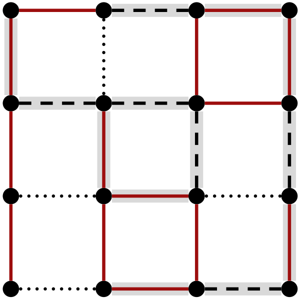
Switching z alongside this cut

- only affects cut edges,
- transforms all 1-edges into 0-edges,
- may alter values of fractional edges.



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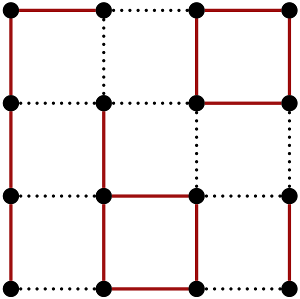
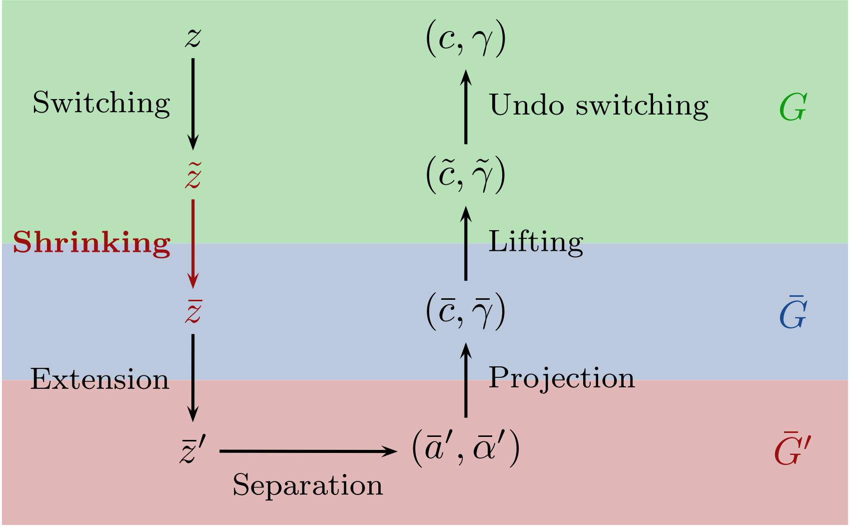
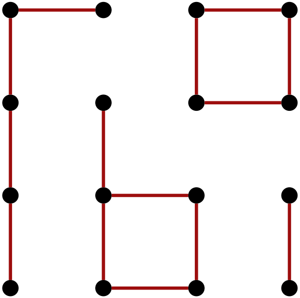


Switched vector $\tilde{z}$ has only fractional and 0-edges.

Shrink 0-edges.

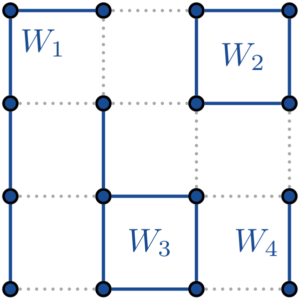


Consider the graph G_0 induced by the 0-edges of the switched vector $\tilde{z}$.



Consider the graph G_0 induced by the 0-edges of the switched vector $\tilde{z}$.

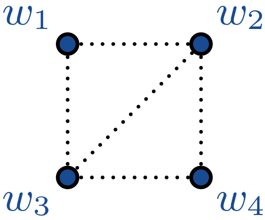
1. determine connected components of G_0,



Shrinking

Consider the graph G_0 induced by the 0-edges of the switched vector $\tilde{z}$.

1. determine connected components of G_0,
2. shrink each component to a supernode.



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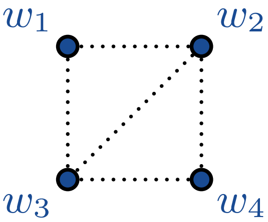
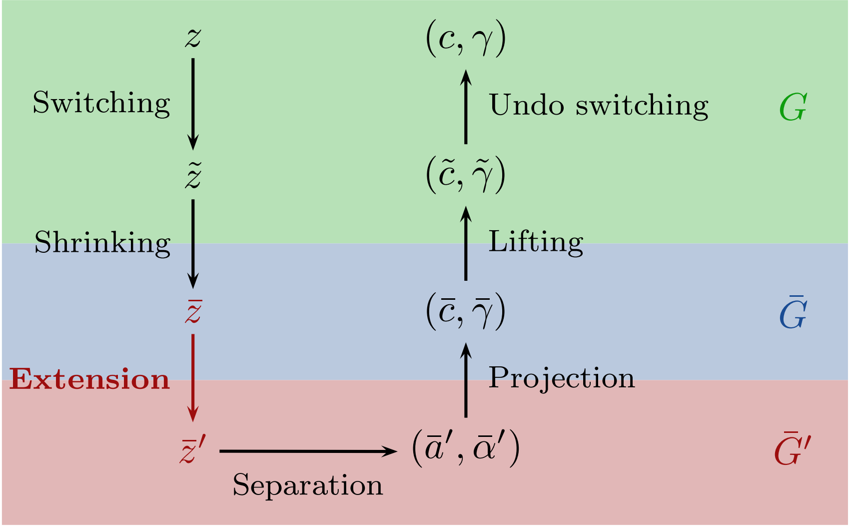
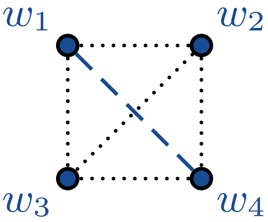


Shrunk vector $\bar{z}$ has only fractional edges. Associated graph $\bar{G}$ may not be complete.

Introduce artificial values for missing edges.



Assign artificial values to missing edges.
Extended vector $\bar{z}'$ shall be in $\mathsf{M}(\bar{G}')$.



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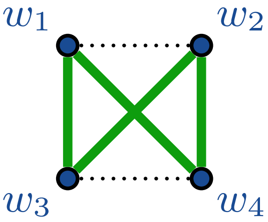
Idea

New cycles in the extended graph consist of



Extension

Assign artificial values to missing edges.
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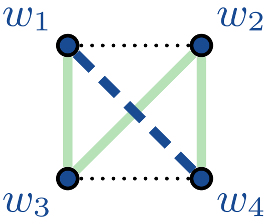


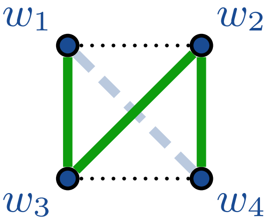
Idea

New cycles in the extended graph consist of an artificial edge

Assign artificial values to missing edges.
Extended vector $\bar{z}'$ shall be in $\mathsf{M}(\bar{G}')$.

Idea

New cycles in the extended graph consist of an artificial edge and a connecting path.



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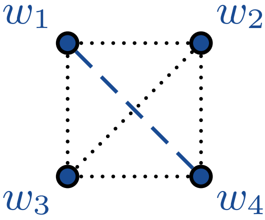


Idea

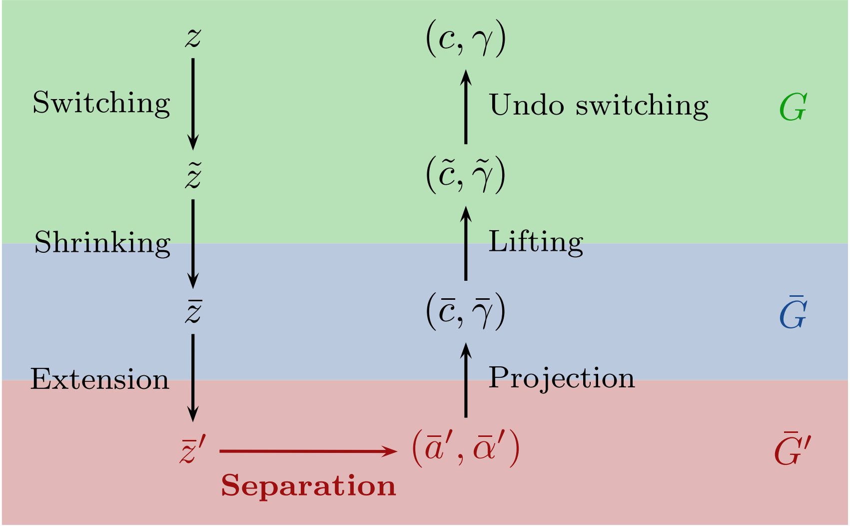
New cycles in the extended graph consist of an artificial edge and a connecting path.

Feasible artificial values

Use shortest-path algorithm to compute range $[\xi_l, \xi_u] \subseteq [0, 1]$ of feasible artificial values for each missing edge.

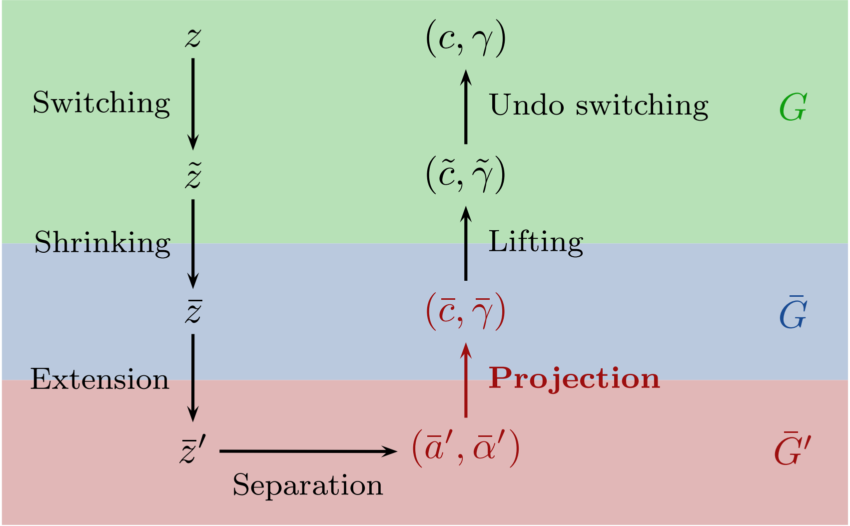
$$\begin{aligned} \xi_l &:= \max \{\bar{z}(F) - \bar{z}(P \setminus F) - |F| + 1 \mid F \subseteq P, |F| \text{ odd}, P \text{ connecting path}\}, \\ \xi_u &:= \min \{-\bar{z}(F) + \bar{z}(P \setminus F) + |F| \mid F \subseteq P, |F| \text{ even}, P \text{ connecting path}\}. \end{aligned}$$

Separate extended vector using techniques for complete graphs.



Outline of the shrinking approach

Project out coefficients related to missing edges.



Projection

Separation $\longrightarrow$ violated inequality $\bar{a}'^T \bar{x}' \leq \bar{\alpha}'$ (denoted $(\bar{a}', \bar{\alpha}')$). Missing edges may have non-zero coefficients!

$$(\cdots \quad \bar{a}'_e \quad \cdots \quad \bar{a}'_f \quad \cdots, \bar{\alpha}')$$

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$$\begin{array}{rl} & (\cdots \;\; \bar{a}'_e \;\; \cdots \;\; \bar{a}'_f \;\; \cdots,\; \bar{\alpha}') \\ + & (\cdots -\bar{a}'_e \;\; \cdots \;\; \cdots \;\; \cdots,\; \bar{\beta}'_1) \\ + & (\cdots \;\; \cdots \;\; \cdots -\bar{a}'_f \;\; \cdots,\; \bar{\beta}'_2) \end{array}$$

Project out artificial non-zero coefficients

- add multiples of suited valid inequalities,
- odd-cycle inequalities defining the bounds ξ_l, ξ_u are possible candidates.

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$$
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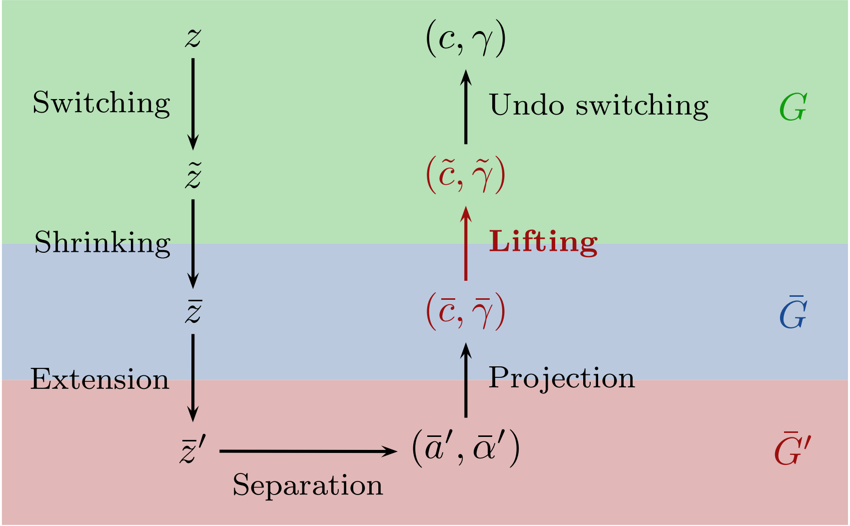
+ & (\cdots \;\; \cdots \;\; \cdots -\bar{a}'_f \;\; \cdots, \; \bar{\beta}'_2) \\
\hline
= & (\cdots \;\; 0 \;\; \cdots \;\; 0 \;\; \cdots, \; \bar{\gamma})
\end{array}
$$

Project out artificial non-zero coefficients

- add multiples of suited valid inequalities,
- odd-cycle inequalities defining the bounds ξ_l, ξ_u are possible candidates.

In the projected inequality all coefficients of missing edges are zero. Truncation $\longrightarrow$ $(\bar{c}, \bar{\gamma})$.

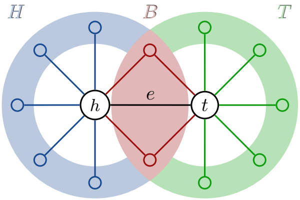
Lift inequality.



Required information

When shrinking edge $e = (h, t)$ store sets:

- $H = \{$ exclusive neighbors of $h\}$,
- $T = \{$ exclusive neighbors of $t\}$,
- $B = \{$ common neighbors of h and $t\}$.



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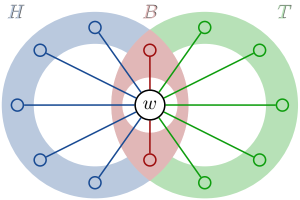

Lift the inequality

- distribute coefficients of edges of the shrunk graph

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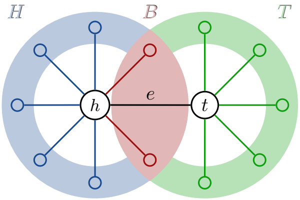

Lift the inequality

- distribute coefficients of edges of the shrunk graph to edges of the original graph w.r.t. above sets,

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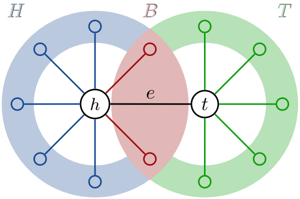
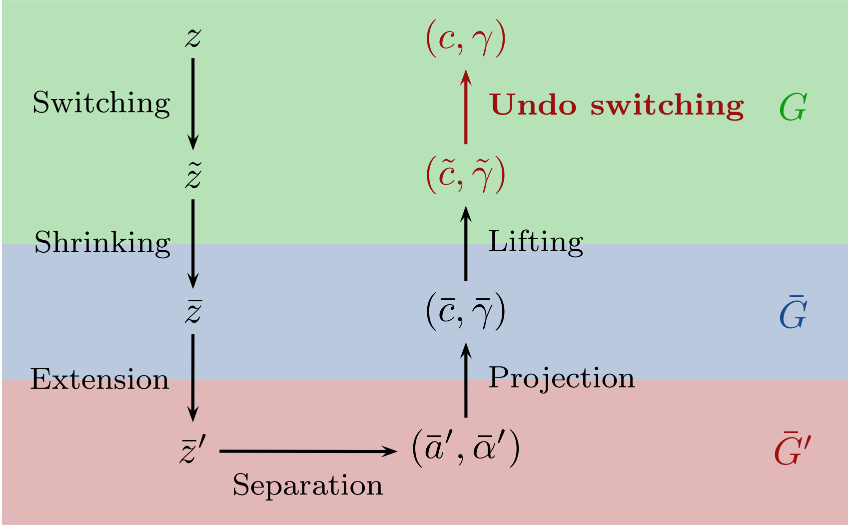


Lift the inequality

- distribute coefficients of edges of the shrunk graph to edges of the original graph w.r.t. above sets,
- edge e gets coefficient $-\min\{\sum_{v\in T} |\bar{c}_{wv}|, \sum_{v\in H} |\bar{c}_{wv}|\}$.

Switch back inequality.



Outline

First computational experiments

Implementation in C++ using the B&C framework ABACUS.

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Examined problem classes

2d/3d torus graphs related to spin glass problems. Edge weights:

- ± 1 (probability 0.5 for positive weight),
- Gaussian distributed.

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Benchmark settings

Compute B&C root bound with following separator settings:

1. **no shrink**: SHOC, 4-cycles, exact odd-cycles.
2. **bike**: additional bicycle-p-wheels on shrunk graph.
3. **target**: additional target cuts on shrunk graph.

Tentative results compared to “no shrink” setting

2d torus graphs

Setting	plus-minus			gauss		
	Gap	#LPs	Time	Gap	#LPs	Time
bike	⋆	▼ 30 %	▼ 20 %	⋆	▼ 79 %	▼ 58 %
target	⋆	▼ 30 %	▼ 17 %	⋆	▼ 79 %	▼ 44 %

3d torus graphs

Setting	plus-minus			gauss		
	Gap	#LPs	Time	Gap	#LPs	Time
bike	▼ 0.8 %	▲ 39 %	▲ 8233 %	▼ 6.3 %	▲ 13 %	▲ 470 %
target	▼ 0.1 %	▼ 8 %	▲ 296 %	▼ 4.4 %	▲ 5 %	▲ 90 %

Conclusion

Separation method for max-cut problems based on graph shrinking:

- enables transfer of separation techniques from dense/complete graphs to sparse graphs,
- shows potential to improve solvability of max-cut problems at least for certain problem classes.

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Future work

- identify and eliminate bottlenecks,
- test different perturbations on ± 1-torus graphs,
- test different shrinking orders (e.g. randomization),
- develop alternative to usage of cycle polytope M(G).

Acknowledgment

- **Prof. G. Reinelt** (University of Heidelberg)
- **Dr. M. Oswald** (University of Heidelberg)
- **Prof. G. Rinaldi** (IASI Rome)
- **Prof. M. Jünger** (University of Cologne)

Thank you!

Thank you for your attention!