Selected Problems in Discrete Optimization

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Overview

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Linear combinatorial optimization problem

Definition (Linear combinatorial optimization problem)

Let E be a finite set, $\mathcal{I} \subseteq 2^E$ the subset of feasible solutions and $c\colon E \to \mathbb{R}$ a function. The task is to determine a set $I^* \in \mathcal{I}$ such that $c(I^*) = \sum_{e \in I^*} c(e)$ is minimal (maximal).

Example (Traveling Salesman Problem)

We are given n points in the euclidean plane and want to determine a closed walk through all the points that visits each point exactly once and is as short as possible, i.e. a shortest tour.

$$E := \text{set of connections between two points}$$

$$\mathcal{I} := \text{sets of connections building a tour}$$

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Some examples for applications

Shortest paths
Compute a shortest connection between two points, e.g. route planning.

Network design
Connect a set of nodes with a communication network.

Cutting problems
Minimize the amount of waste when cutting paper webs.

Allocation of frequencies in a cellular phone network
Assign frequencies to the different antennas such that interferences are minimized and the coverage is maximized.

Aircrew scheduling
Find a feasible and cost-efficient allocation of available crews to the different flights.

Overview

Graph search

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Let $G = (V, E)$ be an undirected graph. Determine a way to systematically visit all nodes.

Breadth-First-Search and Depth-First-Search

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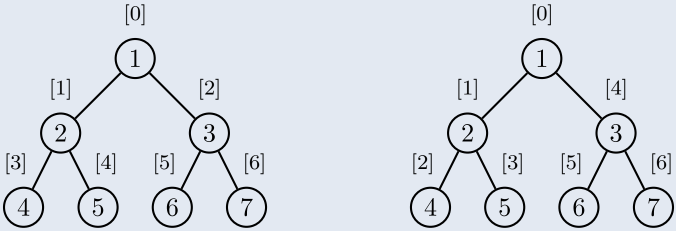

(a) Breadth-First-Search (b) Depth-First-Search

Topological sort

Topological sort problem

Directed acyclic graphs are often used to indicate precedences among certain events. Determine a linear ordering of the events with respect to the given precedences, i.e. a so-called topological sort.

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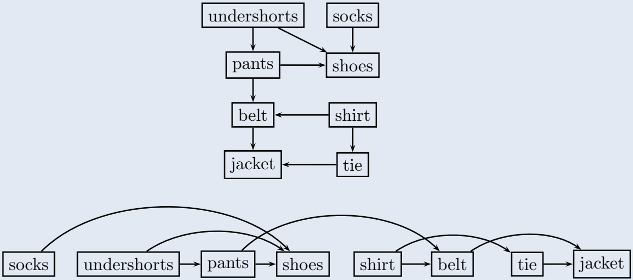

(Cormen, Leiserson, Rivest. *Introduction to algorithms*)

Overview

Minimum spanning trees

Minimum spanning tree problem

Let $G = (V, E)$ be an undirected weighted graph with weights c_e, $e \in E$. Determine a connected acyclic subgraph $T = (V, E')$, $E' \subseteq E$, with minimum weight.

Example (Power grid)

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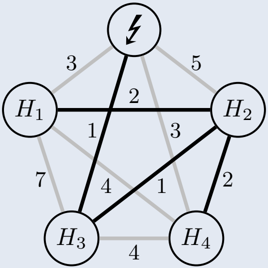

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Shortest paths

Shortest path problem

Let $D = (V, A)$ be a weighted digraph with weights c_e, $e \in A$, and let $s, t \in V$ be two nodes. Determine an (s, t)-path within D with minimum weight.

Example (Route planning)

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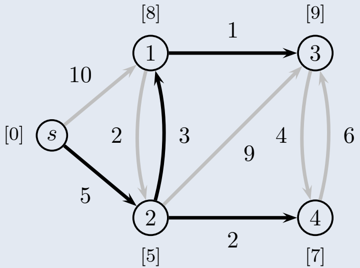

Bottleneck problem

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Let $D = (V, A)$ be a weighted digraph with weights c_e, $e \in A$, and let $s, t \in V$ be two nodes. Determine an (s, t)-path within D whose shortest edge has maximum length, i.e. solve

$$\max_{P\ (s,t)\text{-path}} \min_{e \in P} c_e.$$

Example (Large goods vehicle routing)

Consider a large goods vehicle (LGV). When planning a transport, the overall length of the route is of minor importance. Instead, it is vital that the lowest vertical clearance of a bridge on the way is maximized over all possible routes.

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Matchings on bipartite graphs

Perfect matching problem on bipartite graphs

Let $K_{n,n} = (V_1 \uplus V_2, E)$ be the complete bipartite graph with $2n$ nodes and edge weights c_{ij}, $i \in V_1$, $j \in V_2$. Determine a perfect matching on $K_{n,n}$ with minimum weight.

Example (Marriage problem)

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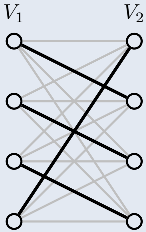
Example (Marriage problem)

$$C := \begin{pmatrix} 3 & 5 & 8 & 4 \\ 9 & 9 & 9 & 6 \\ 6 & 7 & 8 & 4 \\ 3 & 8 & 9 & 2 \end{pmatrix}$$



Overview

1. Motivation

2. Polynomial Problems
   - Basic graph algorithms
   - Minimum spanning trees
   - Shortest paths
   - Matchings
   - Maximum flows
   - Minimum cuts
   - Minimum cost flows
   - Linear programming

3. $\mathcal{NP}$-hard problems

Maximum flows

Maximum (s, t)-flow problem

Let $D = (V, A)$ be a weighted digraph with edge capacities c_e, $e \in A$, and let $s, t \in V$ be two nodes. Determine a maximum (s, t)-flow.

Example (Maximal fluid quantity in pipeline networks)

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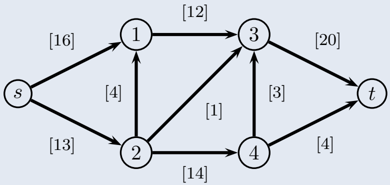
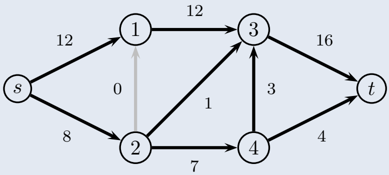


(a) pipeline network with capacities



(b) maximum fluid transfer

Data integrity

Data integrity problem

Let $D \in \mathbb{N}^{p \times q}$ be a matrix. Denote with $r_i, c_j > 0$ the i-th row sum and the j-th column sum respectively. Given these sums and a set of known entries Y, determine all entries whose value can be extracted from this information.

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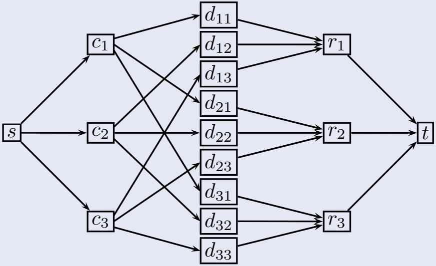
Example

	1	4	1
1	0	1	0
2	0	2	0
3	1	1	1



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Minimum cuts

Minimum cut problem

Let $G = (V, E)$ be a connected undirected weighted graph with weights $c_e > 0$, $e \in E$. Determine a node set $\emptyset \neq W \subsetneq V$ that minimizes the weight sum of all edges with exactly one node in W:

$$\min_{\emptyset \neq W \subsetneq V} c\big(\delta(W)\big).$$

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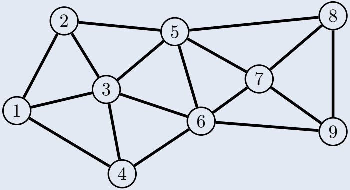

At least three edges have to be removed in order to destroy the connectivity of the network.

Overview

Minimum cost flows

Minimum cost flow problem

Let $D = (V, A)$ be a digraph with edge capacities c_e, $e \in A$, edge costs w_e, $e \in A$, and node balances b_u, $u \in V$. Determine a flow with minimum costs that fulfills all node balances.

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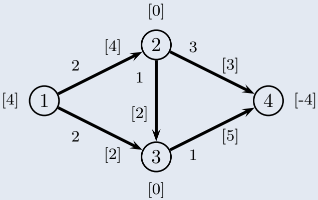
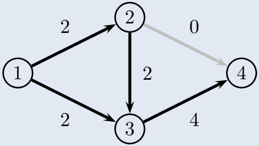


(a) transportation network



(b) minimum cost flow

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Linear programming

Standard LP-formulation

Let $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^m$. The standard formulation of a linear program is as follows:

$$\begin{array}{lrcl} \min & c^\top x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

Remark

Although the running time of the widely used Simplex method is exponential in worst case, linear programming itself is polynomial ($\rightarrow$ Ellipsoid method).

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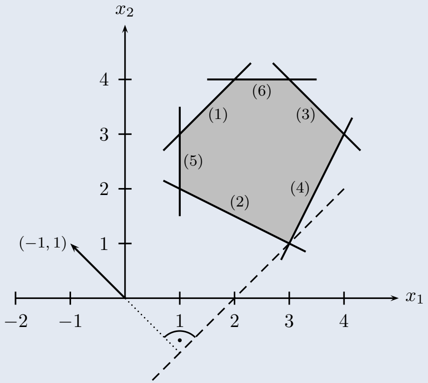
Linear programming (cont.)

Example

$$
\begin{array}{lrcrcrl}
\text{min} & -x_1 & + & x_2 & & & \\
\text{s.t.} & -x_1 & + & x_2 & \leq & 2 & (1) \\
 & -x_1 & - & 2x_2 & \leq & -5 & (2) \\
 & x_1 & + & x_2 & \leq & 7 & (3) \\
 & 2x_1 & - & x_2 & \leq & 5 & (4) \\
 & -x_1 & & & \leq & -1 & (5) \\
 & & & x_2 & \leq & 4 & (6)
\end{array}
$$



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Integer and mixed integer programming

IP- and MIP-formulation

Let $A \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{n\times l}$, $b \in \mathbb{R}^n$, $d \in \mathbb{R}^l$, $c \in \mathbb{R}^m$. The standard formulations of an integer program and a mixed integer program respectively are as follows:

$$\begin{array}{lrcl} \min & d^\top y & & \\ \text{s.t.} & Dy & \leq & b \\ & y & \in & \mathbb{Z}^l_+ \end{array} \qquad \begin{array}{lrcrcl} \min & c^\top x & + & d^\top y & & \\ \text{s.t.} & Ax & + & Dy & \leq & b \\ & x & & & \geq & 0 \\ & & & y & \in & \mathbb{Z}^l_+ \end{array}$$

Comparison of the feasible regions of LP, IP and MIP

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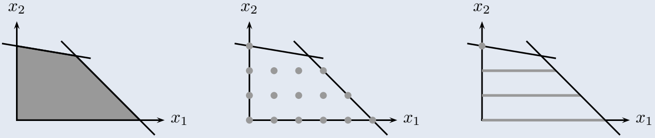

(a) LP (b) IP (c) MIP

Overview

Traveling salesman problem

Traveling salesman problem (TSP)

A salesman has to visit $n-1$ cities. Find a cost-efficient tour through all cities that starts and ends in his hometown and visits each of the cities exactly once.

IP-formulation

Let $K_n = (V, E)$ be the complete graph with n nodes and edge weights c_e, $e \in E$. A possible IP-formulation of the TSP is the following:

$$\begin{array}{lllll} \min & \sum_{e\in E} c_e x_e & & & \\ \text{s.t.} & \sum_{e\in\delta(v)} x_e & = & 2, & \forall\, v \in V \\ & \sum_{e\in\delta(U)} x_e & \geq & 2, & \forall\, 2 \leq |U| \leq \left\lfloor \frac{|V|}{2} \right\rfloor \\ & x_e & \in & \{0,1\}, & \forall\, e \in E \end{array}$$

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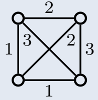
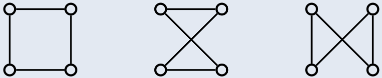
Traveling salesman problem (cont.)

Example

For K_n we have $\frac{(n-1)!}{2}$ different tours. Consider K_4 with the following edge weights



The three possible tours are



with weights $7, 8$ and 9 respectively. Hence, the optimal tour is the first one with minimal weight 7.

Thank you!

We would like to thank you for your interest and your attention!